PMM U.S.S.R.,Vol.47,No.2,pp. 220-226,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.3

ON BOUNDARY CONDITIONS IN THE THEORY OF PIEZOCERAMIC SHELLS POLARIZED ALONG COORDINATE LINES*

N.N. ROGACHEVA

An approximate method /1/ is used for converting three-dimensional boundary conditions to two-dimensional boundary conditions for piezoceramic shells polarized in advance along one of sets of coordinate lines of the median surface for various methods of fixing edges covered with electrodes and free of these. More precise equations of piezoceramic shells with the polarization considered here are obtained by the same method in /2/.

1. We assume that, as in the case of non-electric shells /1/, the complete electroelastic state can be represented as the sum of two electroelastic states, one of which varies comparatively slowly along the median surface coordinate lines and is defined by equations of the theory of piezoceramic shells /2/ (internal electroelastic state), and the other electroelastic state, that of the boundary layer, is rapidly attenuated in the perpendicular direction to the edge, and is defined by three-dimensional equations.

At the shell rim the internal electroelastic state interacts with the boundary layer under imposed boundary conditions.

All the notation used here is the same as in /2/.

We refer the shell to a triorthogonal system of coordinates $\alpha_1, \alpha_2, \gamma$ where α_1-, α_2- are lines that coincide with the lines of curvature of the median surface, and the γ-lines are orthogonal to them.

Since with preliminary polarization along the α_2-lines the directions of α_1 and α_2 are not the same, we consider separately the edges $\alpha_1 = \alpha_{10}$ and $\alpha_2 = \alpha_{20}$.

We introduce into equations of the boundary layer instead of the symmetric $\sigma_{\mu\rho}$ (μ, ρ = 1, 2, 3) the asymmetric tensor of stresses, and instead of the electrical induction vector $\mathbf{D}^*$ and of the electric field strength vector $\mathbf{E}^*$ introduce vectors $\mathbf{D}$ and $\mathbf{E}$ by formulas

$$S_{ii} = \left(1 + \frac{\gamma}{R_j}\right)\sigma_{ii}, \quad S_{ij} = \left(1 + \frac{\gamma}{R_i}\right)\sigma_{ij} \tag{1.1}$$

$$S_{i3} = S_{3i} = \left(1 + \frac{\gamma}{R_j}\right)\sigma_{3i}, \quad S_{33} = \left(1 + \frac{\gamma}{R_1}\right)\left(1 + \frac{\gamma}{R_2}\right)\sigma_{33}$$

$$D_i = \left(1 + \frac{\gamma}{R_j}\right)D_i^*, \quad D_3 = \left(1 + \frac{\gamma}{R_1}\right)\left(1 + \frac{\gamma}{R_2}\right)D_3^*$$

$$E_i = \left(1 + \frac{\gamma}{R_i}\right)E_i^* \quad (i \neq j = 1, 2)$$

Let the shell edge coincide with the line $\alpha_i = \alpha_{i0}$.

We carry out in the three-dimensional equations of piezoelasticity the following substitution of variables:

$$\alpha_i - \alpha_{i0} = R\eta^1\xi_i, \quad \alpha_j = R\eta^t\xi_j, \quad \gamma = R\eta^1\zeta \ (i \neq j = 1, 2) \tag{1.2}$$

where t is the index of variablity of inner electroelastic state, and η is the ratio of the shell half-thickness h to the characteristic dimension R. This substitution means that we are seeking the three-dimensional electroelastic state with variability index equal unity in the direction normal to the edge and along the normal to the median surface of the shell, and which has a much smaller variability index along the edge.

By analogy with the theory of boundary layer of nonelectric shells we represent the boundary layer as a sum of plane and antiplane boundary layers (it will be seen from the equations below that here the terms "plane" and "antiplane" are purely formal, and can be explained only in relation to mechanical quantities).

---

*Prikl.Matem.Mekhan.,Vol.47,No.2,pp.263-270,1983

2. On the edge $\alpha_1 = \alpha_{10}$ for the quantities of the antiplane and the plane boundary layers we have the following asymptotics:

$$(S_{12}{}^k, S_{21}{}^k, S_{23}{}^k, v_2{}^k/h, \psi^k/h, E_1{}^k, E_3{}^k, D_1{}^k, D_3{}^k) = \eta^r (S^k_{12*}, S^k_{21*}, S^k_{23*}, V^k_{2*}, \psi_*{}^k, E^k_{1*}, E^k_{3*}, D^k_{1*}, D^k_{3*}) \tag{2.1}$$

$$E_2{}^k = \eta^{1-t+r} E^k_{2*}$$

$$(S_{11}{}^k, S_{22}{}^k, S_{33}{}^k, v_1{}^k/h, v_3{}^k/h, D_2{}^k) = \eta^{1-t-r} (S^k_{11*}, S^k_{22*}, S^k_{33*}, S^k_{13*}, V^k_{1*}, V^k_{3*}, D^k_{2*}) \tag{2.2}$$

The validity of these asymptotics is supported by that in the initial approximations consistent systems of equations of plane and antiplane boundary layer are obtained.

Formulas (2.1) and (2.2) mean that instead of each unknown quantity is introduced the respective quantity with an asterisk multiplied by η in some power, for example

$$S_{12}{}^k = \eta^r S^k_{12*}$$

The powers of η are selected so that the quantities with asterisks are of the same order.

The superscript k in (2.1) and (2.2) must be replaced below by either a or b for quantities of antiplane (let us set $r = 0$) and plane $(r = 1 - t)$ boundary layers, respectively.

Substituting the antiplane boundary layer asymptotics into the three-dimensional equations of piezoelasticity and carrying out the substitutions of variables (1.2) $(i = 1, j = 2)$, we obtain a system of equations which can be split into two subsystems: the principal and the subsidiary. These subsystems are to be integrated successively: first, from the principal subsystem we determine the quantities (2.1), and then from the equations of the subsidiary subsystem we determine the sought quantities (2.2), assuming the quantities (2.1) to be known. In exactly the same way for the plane boundary layer we obtain the principal subsystem for quantities (2.2) and the subsidiary subsystem for the quantities (2.1).

Let us write down the equations of the principal subsystems for the antiplane and plane boundary layers in the initial approximation

$$\frac{1}{A_{10}}\frac{\partial S^a_{21*}}{\partial \xi_1} + \frac{\partial S^a_{23*}}{\partial \zeta} = 0, \quad \frac{1}{A_{10}}\frac{\partial V^a_{2*}}{\partial \xi_1} - d_{15}E^a_{1*} = s_{44}{}^E S^a_{12*} = s_{44}{}^E S^a_{21*} \tag{2.3}$$

$$\frac{\partial V^a_{2*}}{\partial \zeta} - d_{15}E^a_{3*} = s_{66}{}^E S^a_{23*}, \quad D^a_{1*} = \varepsilon_{11}{}^T E^a_{1*} + d_{15}S^a_{12*}$$

$$D^a_{3*} = \varepsilon_{11}{}^T E^a_{3*} + d_{15}S^a_{23*}, \quad \frac{\partial D^a_{3*}}{\partial \zeta} + \frac{1}{A_{10}}\frac{\partial D^a_{1*}}{\partial \xi_1} = 0$$

$$E^a_{3*} = -\frac{\partial \psi_*{}^a}{\partial \zeta}, \quad E^a_{1*} = -\frac{1}{A_{10}}\frac{\partial \psi_*{}^a}{\partial \xi_1}, \quad E^a_{2*} = -\frac{1}{A_2}\frac{\partial \psi_*{}^a}{\partial \xi_2}$$

$$S^a_{23*} = 0, \quad \zeta = \pm 1 \tag{2.4}$$

$$\psi_*{}^a = 0, \quad \zeta = \pm 1 \tag{2.5}$$

$$D^a_{3*} = 0, \quad \zeta = \pm 1 \tag{2.6}$$

$$\frac{1}{A_{10}}\frac{\partial S^b_{11*}}{\partial \xi_1} + \frac{\partial S^b_{13*}}{\partial \zeta} = 0, \quad \frac{1}{A_{10}}\frac{\partial S^b_{31*}}{\partial \xi_1} + \frac{\partial S^b_{33*}}{\partial \zeta} = 0 \tag{2.7}$$

$$s_{13}{}^E S^b_{11*} + s_{33}{}^E S^b_{22*} + s_{13}{}^E S^b_{33*} = 0$$

$$\frac{1}{A_{10}}\frac{\partial V^b_{1*}}{\partial \xi_1} = s_{11}{}^E S^b_{11*} + s_{13}{}^E S^b_{22*} + s_{12}{}^E S^b_{33*}$$

$$\frac{\partial V^b_{3*}}{\partial \zeta} = s_{12}{}^E S^b_{11*} + s_{13}{}^E S^b_{22*} + s_{11}{}^E S^b_{33*}$$

$$\frac{\partial V^b_{1*}}{\partial \zeta} + \frac{1}{A_{10}}\frac{\partial V^b_{3*}}{\partial \xi_1} = s_{66}{}^E S^b_{13*}$$

$$D^b_{2*} = d_{31}S^b_{11*} + d_{33}S^b_{22*} + d_{31}S^b_{33*}$$

$$S_{13*}{}^b = 0, \quad S^b_{33*} = 0, \quad \zeta_1 = \pm 1 \tag{2.8}$$

$$A_{10} = A_1, \quad A_{20} = A_2, \quad \xi_1 = 0$$

where v_1, v_2, v_3 are three-dimensional displacements, ψ is the electrical potential, and $s_{11}{}^E$, $s_{12}{}^E$, $s_{13}{}^E$, $s_{33}{}^E$, $s_{44}{}^E$, $s_{66}{}^E$, d_{31}, d_{33}, d_{15}, $\varepsilon_{11}{}^T$, $\varepsilon_{33}{}^T$ are the elastic and electrical constants.

Assuming that mechanical and electrical surface loads are taken into account in the integration of equations of internal electroelastic state, we obtain for the boundary layer the homogeneous conditions (2.4)–(2.6) and (2.8). Conditions (2.5) relate to face surfaces and conditions (2.6) must be satisfied on the face surfaces free of electrodes.

3. For deriving the equations of boundary layer on the edge $\alpha_2 = \alpha_{20}$ we substitute in three-dimensional equations of piezoelasticity (1.2) $(i = 2,\ j = 1)$, and assume for the unknown quantities the following asymptotics:

$$(S_{12}{}^k, S_{21}{}^k, S_{13}{}^k, v_1{}^k/h, D_1{}^k) = \eta^r (S_{12*}^k, S_{21*}^k, S_{13*}^k, V_{1*}^k, D_{1*}^k) \tag{3.1}$$

$$(S_{11}{}^k, S_{22}{}^k, S_{33}{}^k, S_{23}{}^k, v_2{}^k/h, v_3{}^k/h, \psi^k/h, E_2{}^k, E_3{}^k, D_2{}^k, D_3{}^k) = \tag{3.2}$$
$$\eta^{1-t-r} (S_{11*}^k, S_{22*}^k, S_{33*}^k, S_{23*}^k, V_{2*}^k, V_{3*}^k, \psi_*{}^k, E_{2*}^k, E_{3*}^k, D_{2*}^k, D_{3*}^k),$$
$$E_1{}^k = \eta^{2-2t-r} E_{1*}^k$$

For the antiplane and plane boundary layers we set the number r equal to 0 and $1-t$ respectively. The principal subsystems of the antiplane and plane boundary layer in the initial approximation at the edge $\alpha_2 = \alpha_{20}$ are

$$\frac{1}{A_{20}}\frac{\partial S_{12*}^a}{\partial \xi_2} + \frac{\partial S_{13*}^a}{\partial \zeta} = 0, \quad \frac{1}{A_{20}}\frac{\partial V_{1*}^a}{\partial \xi_2} = s_{44}{}^E S_{12*}^a = s_{44}{}^E S_{21*}^a \tag{3.3}$$
$$\frac{\partial V_{1*}^a}{\partial \zeta} = s_{66}{}^E S_{13*}^a, \qquad D_{1*}^a = d_{15} S_{12*}^a$$
$$S_{13*}^a = 0, \quad \zeta = \pm 1 \tag{3.4}$$
$$\frac{1}{A_{20}}\frac{\partial S_{22*}^b}{\partial \xi_2} + \frac{\partial S_{23*}^b}{\partial \zeta} = 0, \quad \frac{1}{A_{20}}\frac{\partial S_{32*}^b}{\partial \xi_2} + \frac{\partial S_{33*}^b}{\partial \zeta} = 0 \tag{3.5}$$
$$s_{11}{}^E S_{11*}^b + s_{13}{}^E S_{22*}^b + s_{12}{}^E S_{33*}^b + d_{31} E_{2*}^b = 0$$
$$\frac{1}{A_{20}}\frac{\partial V_{2*}^b}{\partial \xi_2} = s_{13}{}^E S_{11*}^b + s_{33}{}^E S_{22*}^b + s_{13}{}^E S_{33*}^b + d_{33} E_{2*}^b$$
$$\frac{\partial V_{3*}^b}{\partial \zeta} = s_{12}{}^E S_{11*}^b + s_{13}{}^E S_{22*}^b + s_{11}{}^E S_{33*}^b + d_{31} E_{2*}^b$$
$$\frac{1}{A_{20}}\frac{\partial V_{3*}^b}{\partial \xi_2} + \frac{\partial V_{2*}^b}{\partial \zeta} = s_{66}{}^E S_{23*}^b + d_{15} E_{3*}^b$$
$$D_{3*}^b = \varepsilon_{11}{}^T E_{3*}^b + d_{15} S_{23*}^b$$
$$D_{2*}^b = \varepsilon_{33}{}^T E_{2*}^b + d_{31} S_{11*}^b + d_{33} S_{22*}^b + d_{31} S_{33*}^b$$
$$\frac{\partial D_{3*}^b}{\partial \zeta} + \frac{1}{A_{20}}\frac{\partial D_{2*}^b}{\partial \xi_2} = 0$$
$$E_{3*}^b = -\frac{\partial \psi_*{}^b}{\partial \zeta}, \quad E_{2*}^b = -\frac{1}{A_{20}}\frac{\partial \psi_*{}^b}{\partial \xi_2}, \quad E_{1*}^b = -\frac{1}{A_{10}}\frac{\partial \psi_*{}^b}{\partial \xi_1}$$
$$S_{23*}{}^b = 0, \quad S_{33}{}^b = 0, \quad \zeta = \pm 1 \tag{3.6}$$
$$\psi_*{}^b = 0, \quad \zeta = \pm 1 \tag{3.7}$$
$$D_{3*}{}^b = 0, \quad \zeta = \pm 1 \tag{3.8}$$
$$A_{10} = A_1, \quad A_{20} = A_2, \quad \xi_2 = 0$$

4. Consider a shell whose face surfaces are free of electrodes. The electroelasticity relations were obtained in /2/ with an accuracy to quantities of order

$$\varepsilon = O(\eta^{2-2t}) \tag{4.1}$$

Let us derive the boundary conditions of the theory of piezoceramic shells with the same accuracy.

Let the shell edge $\alpha_2 = \alpha_{20}$ covered with electrodes be rigidly fixed. On the edge surface the electric potential is equal V which depends on time only. The three-dimensional boundary conditions are as follows:

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \quad \psi = V \tag{4.2}$$

We represent each of the quantities (4.2) in the form of three terms, the first of which is determined by the two-dimensional equations of the theory of piezoceramic shells, and the second and third are determined by the equations of antiplane and plane boundary layers, respectively. For quantities of internal electroelastic state we use the asymptotics and the expansions of unknown quantities in coordinate γ given in /2/, for quantities of the boundary layer we take into account asymptotic expansions (3.1) and (3.2). As a result, conditions (4.2) can be represented in the form

$$\eta^{t}(v_{1,0}+\eta^{1-2t+c}\zeta v_{1,1})+\eta^{1+\alpha}RV_{1*}^{a}+\eta^{2-t+\beta}RV_{1*}^{b}=0 \tag{4.3}$$
$$\eta^{t}(v_{2,0}+\eta^{1-2t+c}\zeta v_{2,1})+\eta^{2-t+\alpha}RV_{2*}^{a}+\eta^{1+\beta}RV_{2*}^{b}=0$$
$$\eta^{c}(v_{3,0}+\eta^{1-c}\zeta v_{3,1})+\eta^{2-t+\alpha}RV_{3*}^{a}+\eta^{1+\beta}RV_{3*}^{b}=0$$
$$\eta^{t}\psi_{,0}+\eta^{2-t+\alpha}R\psi_{*}{}^{a}+\eta^{1+\beta}R\psi_{*}{}^{b}=\eta^{t}V$$

The quantities of internal electroelastic state are determined by the inhomogeneous equations which take into account the mechanical and electrical surface loads. Quantities of the antiplane and plane boundary layers are obtained from the homogeneous equations; in connection with this they have multipliers η^{α} and η^{β}. The numbers α and β are selected so as to obtain for boundary layers from (4.3) inhomogeneous conditions for the end faces. The uniquely acceptable values of α, β,

$$\alpha=1-t, \quad \beta=0 \tag{4.4}$$

do not result in contradictions.

Taking into account (4.4) formulas (4.3) are written thus:

$$v_{1,0}+\eta^{1-2t+c}\zeta v_{1,1}+\eta^{2-2t}(RV_{1*}^{a}+RV_{1*}^{b})=0 \tag{4.5}$$
$$v_{2,0}+\eta^{1-2t+c}\zeta v_{2,1}+\eta^{1-t}(\eta^{2-2t}RV_{2*}^{a}+RV_{2*}^{b})=0$$
$$v_{3,0}+\eta^{1-c}\zeta v_{3,1}+\eta^{1-c}(\eta^{2-2t}RV_{3*}^{a}+RV_{3*}^{b})=0$$
$$\psi_{,0}+\eta^{1-t}(\eta^{2-2t}R\psi_{*}{}^{a}+R\psi_{*}{}^{b})=V$$

We consider the last three conditions as the conditions for the end face of the plane boundary layer. As in /1/, we represent the latter as the sum of symmetric and inversely symmetric plane boundary layers.

For the inversely symmetric part of a plane boundary layer we obtain from (4.5) the following end face conditions at the edge $\xi_2=0$:

$$RV_{2*}^{b}+\eta^{-1+t}v_{2,0}=0, \quad RV_{3*}^{b}+\zeta v_{3,1}=0,$$
$$R\chi_{*}{}^{b}+\eta^{-1+t}d_{33}(\psi_{,0}-V)=0$$

where instead of $\psi_{*}{}^{b}$ there is introduced a new unknown functions $\chi_{*}{}^{b}=d_{33}\psi_{*}{}^{b}$ which has the same dimension as $V_{2*}{}^{b}, V_{3*}{}^{b}$.

Let us consider the subsidiary problems 1°–4° with the following conditions at the end face $\xi_2=0$:

$$RV_{2*}{}^{b}+1=0, \quad RV_{3*}{}^{b}=0, \quad R\chi_{*}{}^{b}=0 \qquad \text{(problem 1°)} \tag{4.6}$$
$$RV_{2*}{}^{b}=0, \quad RV_{3*}{}^{b}=0, \quad R\chi_{*}{}^{b}+1=0 \qquad \text{(problem 2°)} \tag{4.7}$$
$$RV_{2*}{}^{b}=0, \quad RV_{3*}{}^{b}+\zeta=0, \quad R\chi_{*}{}^{b}=0 \qquad \text{(problem 3°)} \tag{4.8}$$
$$RV_{2*}{}^{b}=0, \quad RV_{3*}{}^{b}=0, \quad R\chi_{*}{}^{b}=0 \qquad \text{(problem 4°)} \tag{4.9}$$

The first three problems involve integration of equations (3.5) with allowance for conditions at the face surfaces (3.6) and (3.7) and end face conditions (4.6)–(4.8), and the fourth, the integration inhomogeneous equations of the plane boundary layer, in whose right-hand side appear the free terms of order η with allowance for conditions (3.6) and (3.7) and homogeneous conditions at the end faces (4.9). In seeking the solutions of subsidiary problems 1°–4° we shall require that away from the edge the following conditions of decay

$$RV_{2*}{}^{b}=0, \quad RV_{3*}{}^{b}=0, \quad R\chi_{*}{}^{b}=0, \quad \xi_2=-\infty$$

were satisfied.

The theory of the boundary layer is linear, hence the inversely symmetric part of the plane boundary layer can be represented as a linear combination of four subsidiary problems with multipliers

$$\eta^{-1+t}v_{2,0}, \quad v_{3,1}, \quad \eta^{-1+t}d_{33}(\psi_{,0}-V), \quad 1$$

In the solution constructed in this way the translations and electric potential vanish as $\xi_2=-\infty$. We shall require that away from the edge the stresses and the electrical quantities vanish. On physical considerations and from the St.Venant principle it follows that it is necessary to stipulate that at the edge $\xi_2=0$ the resultant of horizontal forces and the components of the vector of electrical induction D_2 normal to the edge vanish

$$F_1\eta^{-1+t}v_{2,0} + F_2\eta^{-1+t}\, d_{33}(\psi_{,0} - V) + F_3 v_{3,1} + F_4\eta^1 = 0$$
$$B_1\eta^{-1+t}v_{2,0} + B_2\eta^{-1+t}\, d_{33}(\psi_{,0} - V) + B_3 v_{3,1} + B_4\eta^1 = 0$$

where F_1, F_2, F_3, F_4 and B_1, B_2, B_3, B_4 denote the horizontal components of force and the normal to the edge components of the vector of electrical induction at the edge $\xi_2 = 0$ of problems 1°– 4°. Solving these equations for $v_{2,0}$ and $(\psi_{,0} - V)$, we obtain with the accuracy (4.1) the following formulas:

$$v_{2,0} + \eta^{1-t}m_1 v_{3,1} = 0, \quad d_{33}(\psi_{,0} - V) + \eta^{1-t}m_2 v_{3,1} = 0$$
$$m_1 = \frac{F_2B_3 - F_3B_2}{F_1B_2 - F_2B_1}, \quad m_2 = \frac{F_3B_1 - F_1B_3}{F_1B_2 - F_2B_1} \tag{4.10}$$

Considering in the same way the symmetric part of the plane boundary layer we obtain

$$v_{1,0} = 0, \quad v_{3,0} = 0, \quad v_{2,1} = 0 \tag{4.11}$$

The three-dimensional displacements are linked with those of the median surface of shells u_1, u_2, w by formulas

$$v_{1,0} = u_1, \quad v_{2,0} = u_2, \quad v_{3,0} = -w$$

Passing in formulas (4.10) and (4.11) to notation used in the theory of shells and expanding $v_{3,1}$ to formula adduced in /2/, we obtain the following boundary conditions ($\alpha_2 = \alpha_{20}$):

$$u_1 = 0, \quad u_2 + hm_1\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(au_2 - b\psi^{(0)}) = 0$$
$$w = 0, \quad \gamma_2 = 0, \quad \psi^{(0)} - V + hm_2\frac{1}{d_{33}}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(au_2 - b\psi^{(0)}) = 0$$
$$a = s_{12}{}^E n_{12} + s_{13}{}^E n_{22}, \quad b = d_{31} - s_{12}{}^E c_1 - s_{13}{}^E c_2$$

where the terms with coefficients m_1 and m_2 bring in corrections of order η^{1-t} into the boundary conditions in comparison with unity.

The remaining boundary conditions are given without derivation.

The hinged edge ($\alpha_2 = \alpha_{20}$) is covered by electrodes,

$$u_1 = 0, \quad w = 0, \quad G_2 = 0$$
$$T_2 - \left\{hk_{10}\frac{n}{c}\left(\frac{s_{12}{}^E}{d_{31}}T_1 - \frac{2h}{A_2}\frac{\partial\psi^{(0)}}{\partial\alpha_2}\right)\right\} = 0$$
$$\psi^{(0)} - V + n\left(\frac{s_{12}{}^E}{2d_{31}}T_1 - \frac{h}{A_2}\frac{\partial\psi^{(0)}}{\partial\alpha_2}\right) = 0$$
$$\left(c = \frac{d_{31}(s_{12}{}^E - s_{11}{}^E)}{d_{31}{}^2 - s_{12}{}^E\varepsilon_{33}{}^T}\right)$$

To determine the number n in it necessary to solve the homogeneous equations of the plane problem (3.5), (3.6), (3.7) with end face conditions

$$1°.\ S_{22*}^b = 0, \quad RV_{3*}^b = 0, \quad R\chi_*{}^b + 1 = 0$$
$$2°.\ S_{22*}^* = 0, \quad RV_{3*}^b + \zeta = 0, \quad R\chi_*{}^b = 0, \quad (\chi_*{}^b = d_{31}\psi_*{}^b)$$

and then to find the resultant of components normal to the edge surface of the electrical induction A_1 and A_2 vector for problems 1° and 2°, respectively and calculate n using formula $n = A_2/A_1$.

The terms which bring in the boundary conditions corrections of order η^1 are everywhere enclosed in brackets.

Free edge $\alpha_2 = \alpha_{20}$ covered by electrodes,

$$T_2 = 0, \quad S_{12} = 0, \quad G_2 + 3l\frac{h}{A_1}\frac{\partial H_{12}}{\partial\alpha_1} = 0 \tag{4.12}$$
$$N_2 - \frac{1}{A_1}\frac{\partial H_{12}}{\partial\alpha_1} = 0, \quad \psi^{(0)} = V$$
$$\left(l = \int_{-1}^{+1}\zeta\, d\zeta \int_{-\infty}^{0} S_{12*}^a A_{20}\, d\xi_2\right)$$

For calculating l it is necessary to construct the solution of equations of antiplane problem (3.3), (3.4) with end face condition at the edge $\xi_2 = 0$

$$S_{12}^{a} + \zeta = 0$$

For a free edge $\alpha_2 = \alpha_{20}$ without electrodes the mechanical condition (4.12) are maintained and the electrical condition must be replaced by the condition

$$D_2^{(0)} = 0$$

Rigidly fixed edge $\alpha_1 = \alpha_{10}$ covered by electrodes,

$$u_1 + m\frac{h}{A_1}\frac{\partial u_1}{\partial \alpha_1} = 0, \quad u_2 = 0, \quad w = 0, \quad \gamma_1 = 0, \quad \psi^{(0)} = V$$

$$m = \frac{P_2}{P_1}\left(s_{12}^{E} n_{11} + s_{13}^{E} n_{21}\right)$$

where P_1, P_2 are the horizontal components of forces acting on the edge $\alpha_1 = \alpha_{10}$ obtained as the result of integration of homogeneous equations of the plane problem (2.7) with conditions on face surfaces (2.8) and with end face conditions ($\xi_1 = 0$):

$$1^\circ.\ RV_{1*}^{b} + 1 = 0, \quad RV_{3*}^{b} = 0$$

$$2^\circ.\ RV_{1*}^{b} = 0, \quad RV_{3*}^{l} + \zeta = 0$$

Hinged edge $\alpha_1 = \alpha_{10}$ covered by electrodes,

$$T_1 - \left\{hk_{20}\frac{(s_{13}^{E})^2}{2s_{33}^{E}}\, pT_2\right\} = 0, \quad G_1 = 0, \quad u_2 = 0$$

$$w = 0, \quad \psi^{(0)} = V$$

$$\left(p = \int_{-1}^{+1} S_{13*}^{b}\big|_{\xi_1=0}\, d\zeta\right)$$

where p is determined from the solutions of Eqs.(2.7) with conditions (2.8) and end face conditions at $\xi_1 = 0$

$$S_{11*}^{b} = 0, \quad RV_{3*}^{b} + \zeta = 0$$

Free edge $\alpha_1 = \alpha_{10}$ covered by electrodes,

$$T_1 = 0, \quad S_{21} = 0, \quad N_1 - \frac{1}{A_2}\frac{\partial H_{21}}{\partial \alpha_2} = 0 \qquad (4.13)$$

$$G_1 + 3l_1\frac{h}{A_2}\frac{\partial H_{21}}{\partial \alpha_2} = 0, \quad \psi^{(0)} = V$$

$$\left(l_1 = \int_{-1}^{+1} \zeta\, d\zeta \int_{-\infty}^{0} S_{12}^{a} A_{10}\, d\xi_1\right)$$

The number l_1 is obtained from the solutions of the antiplane problem (2.3), (2.4), (2.6) with inhomogeneous end face conditions

$$S_{12*}^{a} + \zeta = 0, \quad R\psi_*^{a} = 0$$

At the free edge $\alpha_1 = \alpha_{10}$ devoid of electrodes the first three conditions of (4.13) are maintained, and the remaining conditions are

$$G_1 + 3l_2\frac{h}{A_2}\frac{\partial H_{21}}{\partial \alpha_2} = 0, \quad \frac{1}{A_1}\frac{\partial \psi^{(0)}}{\partial \alpha_1} = 0 \qquad (4.14)$$

$$l_2 = \int_{-1}^{+1} \zeta\, d\zeta \int_{-\infty}^{0} S_{12}^{a} A_{10}\, d\xi_1$$

For the determination of l_2 it is necessary to solve the antiplane problem (2.3), (2.4), (2.6) with the following end face conditions with $\xi_1 = 0$:

$$S_{12*}^{a} + \zeta = 0, \quad D_{1*}^{a} + d_{15}\zeta = 0 \qquad (4.15)$$

5. Let the face surfaces of the shell be totally covered by electrodes and the edges be free of them. The boundary conditions in that case are:

Free edge $\alpha_1 = \alpha_{10}$ without electrodes,

$$T_1 = 0,\ S_{21} + \frac{H_{21}}{R_2} = 0,\ \ G_1 + 3l_3 \frac{h}{A_2} \frac{\partial H_{21}}{\partial \alpha_2} = 0$$

$$N_1 - \frac{1}{A_2} \frac{\partial H_{21}}{\partial \alpha_2} + \left\{3l_3 \frac{h}{A_2} \frac{\partial (k_2 H_{21})}{\partial \alpha_2}\right\} = 0$$

Free edge $\alpha_2 = \alpha_{20}$ without electrodes,

$$T_2 = 0,\ S_{12} + \frac{H_{12}}{R_1} = 0,\ \ G_2 + 3l \frac{h}{A_1} \frac{\partial H_{12}}{\partial \alpha_1} = 0$$

$$N_2 - \frac{1}{A_1} \frac{\partial H_{12}}{\partial \alpha_1} + \left\{3l \frac{h}{A_1} \frac{\partial (k_1 H_{12})}{\partial \alpha_1}\right\} = 0$$

The formula for calculating l is given in Sect.4.

The number l_3 is calculated by the last formula (4.14) in which l_3 is substituted for l_2 from the solution of Eqs.(2.3) with conditions at the face surfaces (2.4), (2.5) with end faces conditions (4.15).

To determine the constants l, l_1, l_2, l_3 it is necessary by the method of dividing the variables to integrate the system of Eqs.(2.3) and (3.3) with respective conditions on the face surfaces and along the edge. As the result of simple calculations. We obtain

$$l = l_1 = l_2 = l_3 = -0.42\,(s_{66}^E/s_{11}^E)^{1/2}$$

We have thus obtained the boundary conditions for piezoceramic shells pre-polarized along one of the sets of the median plane coordinate lines. For shells with face surfaces entirely covered with electrodes, they are the analog of boundary conditions, free nonelectric shells. For shells having no electrodes on face surfaces, five boundary conditions are obtained on each edge. The effect of boundary layer on inner electroelastic state does manifest itself in the bonndary conditions by the appearance of a number of supplementary terms, the important of which is the Kirchhoff correction for shear stresses. For the determination of other supplementary terms it is necessary to calculate the constants $m_1, \ldots, l_3$ from the solutions of supplementary problems. The terms with these constants bring in the boundary conditions corrections of order $\eta^1, \eta^{1-t}, \eta^{2-2t+c}$ in comparison with unity.

REFERENCES

1. GOL'DENVEIZER A.L., Theory of Elastic Thin Shells. Pergamon Press, Book No.09561, 1961.
2. ROGACHEVA N.N., Equations of state of piezoceramic shells. PMM, Vol.45, No.5, 1981.
3. ULITKO A.F., On the theory of oscillations of piezoceramic bodies. In: Thermal Stresses in Structural Elements. Issue 15. Kiev, NAUKOVA DUMKA, 1975.

Translated by J.J.D.